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Holographic Coulomb Branch Flows with $\mathcal{N} = 1$ Supersymmetry

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We obtain a large, new class of $\mathcal{N} = 1$ supersymmetric holographic flow backgrounds with $U(1)^3$ symmetry. These solutions correspond to flows toward the Coulomb branch of the non-trivial $\mathcal{N} = 1$ supersymmetric fixed point. The massless (complex) chiral fields are allowed to develop vevs that are independent of their two phase angles, and this corresponds to allowing the brane to spread with arbitrary, $U(1)^2$ invariant, radial distributions in each of these directions. Our solutions are "almost Calabi-Yau:" The metric is hermitian with respect to an integrable complex structure, but is not Kähler. The "modulus squared" of the holomorphic $(3,0)$-form is the volume form, and the complete solution is characterized by a function that must satisfy a single partial differential equation that is closely related to the Calabi-Yau condition. The deformation from a standard Calabi-Yau background is driven by a non-trivial, non-normalizable 3-form flux dual to a fermion mass that reduces the supersymmetry to $\mathcal{N} = 1$. This flux also induces dielectric polarization of the $D3$-branes into $D5$-branes.

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1. Introduction

In this paper we continue our study of supersymmetric backgrounds in string theory, and most particularly for holography. In the more traditional compactifications of string theory the internal, or compactifying manifold is either compact, or effectively compact[1]. This makes the task of classifying supersymmetric backgrounds somewhat easier. Indeed, for smooth solutions to IIB supergravity it has been shown that $\mathcal{N} = 1$ supersymmetry in four dimensions places some very stringent constraints on the background fields. For example [1,2], if there is no dilaton then the compactification has to be a warped Calabi-Yau manifold with an imaginary self-dual 3-form flux. However, the proof makes strong use of the square-integrability of the background fields, and is therefore invalid for non-compact or singular backgrounds with non-normalizable fields. This exception is precisely what one wishes to study in holography, where the non-normalizable modes correspond to perturbations of the Lagrangian of the holographic field theory. Thus the study of supersymmetric, holographic flows is precisely a study in the exceptions to the theorem of [1].

There are now very large families of physically interesting non-compact backgrounds with reduced supersymmetry. Many of them are based upon Calabi-Yau backgrounds, but some of them are complex, but not Kähler [3–7]. The family we wish to focus on here is one of the simplest holographic flows: One starts with the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and gives a mass to a single $\mathcal{N} = 1$ chiral multiplet. As is well-known, this perturbed theory preserves $\mathcal{N} = 1$ supersymmetry and has a non-trivial infra-red fixed point [8]. The holographic description of this fixed point, and the flow to it, is also well studied [9–13]. Indeed, there has been recent progress in understanding the underlying geometry in terms of spaces that are almost Calabi-Yau manifolds [7]. This paper further develops this work. The two chiral multiplets that are not given a mass remain massless along the flow and so the complete $\mathcal{N} = 1$ supersymmetric field theory also has a two complex-dimensional Coulomb branch. A three-parameter family of flows on this Coulomb branch were studied in [14,15]. One of the parameters was the mass of the chiral multiplet, Φ_3, while the other two were independent vevs of Φ_1 and Φ_2. Since these solutions were based upon gauged supergravity, the vevs of these fields were very restricted and corresponded to brane distributions that spread uniformly in each of these directions. Our purpose here is to analyze solutions in which the branes are allowed to spread with

[1] By effectively compact we mean non-compact, but with normalizable background fields.

arbitrary radial distributions in each of these two directions. This means the solutions will depend upon three variables that correspond to the magnitudes, Φ_j. As in [7], we will be able to characterize our solutions in terms of a deformation of the Calabi-Yau condition.

On a more technical level, we will proceed in the same spirit as [16–21], and use algebraic Killing spinors. In the past, such calculations have involved imposing a high level of symmetry so that the metric functions and fluxes can *only* depend upon two variables. In [19] this led to a result that appeared to depend upon having only two-variables: The solution was determined by a single function $\Psi(u,v)$, but one also needed to construct a conjugate function, $S(u,v)$, that looked like a non-linear analog of the harmonic conjugate of $\Psi(u,v)$. It was thus not clear whether the relatively simple results of [19] were an artifact of the high level of symmetry. In this paper we consider generalizations of the flow of [19] in which there is less symmetry, and the underlying functions depend upon three variables. We will show that the simplicity of the result of [19,7] persists: The non-trivial flow solutions arise from a deformation of the Calabi-Yau condition. Indeed we found that deriving the related Calabi-Yau metric first provided remarkable insights into how to solve the more general problem with non-trivial fluxes considered here. In the process of finding the more general class of solutions we will also simplify and unify the results of [19,7].

In section 2 we will briefly summarize the relevant field theory and use its symmetries to constrain the Ansatz for the holographic theory. In section 3 we will make the complete Ansatz for the holographic background. In section 4 we will find the "wrong solution" in that we will set the fluxes to zero and find the most general Calabi-Yau metric. In section 5 we present the new solutions by showing how the Calabi-Yau equations are successively modified. We then show that the new solutions are "almost Calab-Yau" in that they have an integrable complex structure, the metric is hermitian, there is a holomorphic $(3,0)$-form that squares to the volume form, but the Kähler form is not closed, and thus the metric is not Kähler. In sections 6 and 7 we show how the solutions of [19,7] and [14,15]. are contained in our far more general family. Finally, in section 8 we make some concluding remarks.

2. Some field theory constraints on the holographic dual

The underlying field theory is $\mathcal{N}=4$ super-Yang-Mills theory perturbed by a mass term for one of the three $\mathcal{N}=1$ adjoint chiral superfields. The superpotential has the form:

$$W = \mathrm{Tr}\left(\Phi_3[\Phi_1,\Phi_2]\right) \ + \ \tfrac{1}{2}m\,\mathrm{Tr}\left(\Phi_3^2\right) \ . \tag{2.1}$$

This breaks the supersymmetry to $\mathcal{N}=1$, and the theory flows to a non-trivial $\mathcal{N}=1$ superconformal fixed point in the infra-red [8]. The holographic description of the fixed point and flow may be found in [22,9,10,11] The fields, Φ_1 and Φ_2 remain massless and there is thus a four-dimensional Coulomb branch described in terms of the vevs of Φ_1 and Φ_2. A two-parameter family of holographic flows on this Coulomb branch were studied in [15,14], and a brane-probe study can be found in [12,13]. For the moment we will assume that the vevs of Φ_1 and Φ_2 are zero.

The $\mathcal{N}=4$ theory has an $SO(6)$ $\mathcal{R}$-symmetry, and under the deformation (2.1) this is broken to an $SU(2)$ global symmetry and a $U(1)$ $\mathcal{R}$-symmetry. The $SU(2)$ acts on Φ_1 and Φ_2 as a doublet, while the $\mathcal{R}$ symmetry acts on Φ_1, Φ_2 and Φ_3 with charges $(1/2, 1/2, 1)$ [9]:

$$\Phi_j \to e^{\frac{1}{2}i\alpha}\,\Phi_j\,, \quad j=1,2\,; \qquad \Phi_3 \to e^{i\alpha}\,\Phi_3\,. \tag{2.2}$$

and so both terms in the superpotential (2.1) have $\mathcal{R}$-charge 2, as they must. If we allow the mass, m, to rotate by a phase then we have a further $U(1)$ symmetry under which:

$$\Phi_j \to e^{\frac{1}{2}i\alpha}\,\Phi_j\,, \quad j=1,2\,, \qquad \Phi_3 \to \Phi_3\,, \qquad m \to m\,e^{i\alpha}\,. \tag{2.3}$$

This may, of course, be mixed with the $\mathcal{R}$-symmetry action.

In the holographic dual, the vevs of the scalar fields correspond to directions perpendicular to the branes, and we will represent the three complex directions corresponding to Φ_1, Φ_2 and Φ_3 by three sets of complex polar coordinates: (v,φ_1), (w,φ_2) and (u,φ_3). For later convenience (and to match the conventions of earlier papers), we will make the field theory identifications in which, at least asymptotically:

$$\Phi_1 \sim v\,e^{-i\varphi_1}\,, \qquad \Phi_2 \sim w\,e^{-i\varphi_2}\,, \qquad \Phi_3 \sim u\,e^{+i\varphi_3}\,. \tag{2.4}$$

Thus the $SU(2)$ acts on (v,φ_1) and (w,φ_2), and the $U(1)$ $\mathcal{R}$ symmetry corresponds to:

$$\varphi_j \ \to\ \varphi_j \ -\ \tfrac{1}{2}\,\alpha\,, \quad j=1,2\,; \qquad \varphi_3 \ \to\ \varphi_3 \ +\ \alpha\,. \tag{2.5}$$

Invariance under the $SU(2)\times U(1)_{\mathcal{R}}$ where the $U(1)_{\mathcal{R}}$ is defined by (2.5) means that if any of the fields depend upon the φ_j, then they can only depend upon the sum: $(\varphi_1+\varphi_2+\varphi_3)$.

The IIB theory also has a *ten-dimensional* $\mathcal{R}$ symmetry [23,24,25] that acts on the B field with charge $+1$. The B field is dual to the fermion mass, and this symmetry corresponds to (2.3) in the field theory. In terms of coordinates the latter symmetry is:

$$\varphi_j \ \to\ \varphi_j - \tfrac{1}{2}\,\alpha\,, \quad j=1,2\,, \qquad \varphi_3 \ \to\ \varphi_3\,, \qquad B \to B\,e^{i\alpha}\,. \tag{2.6}$$

One therefore finds that the B field dual to the fermion mass in (2.1) must have the phase dependence:

$$B_{\mu\nu} \ \sim \ e^{i(\varphi_1+\varphi_2+\varphi_3)}\,. \tag{2.7}$$

One can also deduce this result directly from linearization of the supergravity action and using the fact that the fermion masses are dual to the lowest modes of the tensor gauge field, $B_{\mu\nu}$. This is also consistent with the results of [19,7].

From [26,19,7] we also know that the family of flows we seek has a complex structure that matches the intuitive complex structure provided by the field theory. Moreover, by a suitable choice of the B-field gauge, we may take the holographic dual of the fermion mass term to be a B field of holomorphic type $(2,0)$. In addition we also know that for these flows the dilaton background is trivial.

In this paper we want to investigate the Coulomb branch of the flow. We are therefore going to allow Φ_1 and Φ_2 to develop vevs. However, to keep things manageable, we are going to assume that these vevs are invariant under the $U(1)^2 \subset SU(2)\times U(1)_{\mathcal{R}}$. That is, the branes can spread in the (v,w) directions, but will only be allowed to do so in a manner that is independent of (φ_1,φ_2). Thus the metric and all the background fields will have a $U(1)^3$ invariance, but will be allowed to depend arbitrarily upon (u,v,w). It is convenient to represent the $U(1)$ symmetries in terms of Lie derivatives. First, there is the residual $U(1)$ subgroup of $SU(2)$ generated by:

$$(\mathcal{L}_1 - \mathcal{L}_2)\,, \tag{2.8}$$

where $\mathcal{L}_j$ denotes the Lie Derivative along the Killing vector defined by translations along φ_j. Then there is the $\mathcal{R}$ symmetry operator:

$$\mathcal{L}_{\mathcal{R}} \ \equiv \ \mathcal{L}_3 - \tfrac{1}{2}(\mathcal{L}_1+\mathcal{L}_2)\,, \tag{2.9}$$

Finally, there is the extra $U(1)$ is given by:

$$-\tfrac{1}{2}\,(\mathcal{L}_1+\mathcal{L}_2) \ + \ Q_{IIB}\,, \tag{2.10}$$

where Q_{IIB} is the IIB $\mathcal{R}$ charge of the field upon which this operator acts.

The Killing spinors that generate the unbroken supersymmetry transformations must transform appropriately under these $U(1)$'s. First, before we turn on the vevs of Φ_1,Φ_2, the supersymmetries must be a $SU(2)$ singlet, and thus:

$$(\mathcal{L}_1 - \mathcal{L}_2)\,\epsilon \ = \ 0\,. \tag{2.11}$$

The operator $\mathcal{L}_{\mathcal{R}}$ must generate the $\mathcal{R}$-symmetry and hence:

$$\mathcal{L}_{\mathcal{R}}\,\epsilon \;\equiv\; (\mathcal{L}_3 - \tfrac{1}{2}(\mathcal{L}_1+\mathcal{L}_2))\,\epsilon \;=\; \Gamma^{1234}\,\epsilon\,. \tag{2.12}$$

The right hand side of this equation is precisely reflects the fact that the $\mathcal{R}$-symmetry rotates the four-dimensional spinor components with charges ± 1 depending upon their helicity. Under the last $U(1)$ one has:

$$\tfrac{1}{2}(\mathcal{L}_1+\mathcal{L}_2)\,\epsilon \;=\; \tfrac{1}{2}\,\epsilon\,, \tag{2.13}$$

where we have used the fact that $Q_{IIB}=\frac{1}{2}$ for the supersymmetry.

There are some signs and ambiguities in the the foregoing prescription. First, the is the sign of the term on the right-hand side of (2.12) depends upon spinor conventions. Secondly, it is not obvious that the action of (2.13) should not be combined with a four-dimensional $\mathcal{R}$ symmetry transformation of the supersymmetry parameter, but it turns out that the ten-dimensional chiral rotation implied by $Q_{IIB}=\frac{1}{2}$ is all that one needs. The complete justification of the foregoing angular dependences of ϵ really comes from the fact that they are required by the solutions of the supersymmetry conditions that we analyze below[2]. Our purpose here is to make the angular behaviour of the Killing spinors more intuitive. Having done this, we have pinned down solution sufficiently to provide a readily solvable Ansatz for the holographic dual in supergravity.

3. The Supergravity Background

3.1. The metric and complex structure

We take the ten-dimensional manifold to have the usual warped product form:

$$ds^2_{10} \;=\; H_0^2(\eta_{\mu\nu}\,dx^\mu\,dx^\nu) \;-\; H_0^{-2}\,ds_6^2\,, \tag{3.1}$$

where ds_6^2 is a hermitian metric on a complex manifold, $\mathcal{M}_6$, transverse to the D3 branes. Following from the field theory, we will parametrize this "internal manifold" by three complex coordinates whose phases, φ_j, $j=1,2,3$, generate a $U(1)^3$ symmetry of the background. The remaining, "radial coordinates," will be denoted by (u,v,w). Following [19,7], it is natural to single out a complex coordinate $z_3 \equiv u\,e^{i\varphi_3}$ that is to be associated

[2] They can also be deduced from the results of [19].

with the directions dual to the massive chiral multiplet. One then fibers the remaining two complex directions over this base. To be more explicit, we take the complex coordinates to be

$$z_1 \;\equiv\; e^{h_1+i\varphi_1}\,, \qquad z_2 \;\equiv\; e^{h_2+i\varphi_2}\,, \qquad z_3 \;\equiv\; u\, e^{i\varphi_3}\,. \tag{3.2}$$

for some functions, $h_j(u,v,w)$, and introduce the holomorphic forms:

$$\begin{aligned}
\omega_1 \;&\equiv dh_1 + i\, d\varphi_1 - (\partial_u h_1)\,\omega_3 \;=\; (\partial_v h_1)\, dv + (\partial_w h_1)\, dw + i\,(d\varphi_1 - u\,\partial_u h_1\, d\varphi_3)\,, \\
\omega_2 \;&\equiv dh_2 + i\, d\varphi_2 - (\partial_u h_2)\,\omega_3 \;=\; (\partial_v h_2)\, dv + (\partial_w h_2)\, dw + i\,(d\varphi_2 - u\,\partial_u h_2\, d\varphi_3)\,, \\
\omega_3 \;&\equiv du + i\, u\, d\varphi_3\,.
\end{aligned} \tag{3.3}$$

We then make the metric Ansatz:

$$ds_6^2 \;=\; A_1\,|\omega_1|^2 \;+\; A_2\,|\omega_2|^2 \;+\; A_3\,|\omega_3|^2 \;+\; A_0\,(\omega_1\,\bar\omega_2 \;+\; \omega_2\,\bar\omega_1)\,. \tag{3.4}$$

where the A_j are, as yet arbitrary functions of (u,v,w). This Ansatz is a natural generalization of the results found in [19,7]. The presence of the cross terms with coefficient A_0, are suggested by the angular terms noted in [14,15].

The complex structure is:

$$J \;=\; A_1\,\omega_1\wedge\bar\omega_1 \;+\; A_2\,\omega_2\wedge\bar\omega_2 \;+\; A_3\,\omega_3\wedge\bar\omega_3 \;+\; A_0\,(\omega_1\wedge\bar\omega_2 \;+\; \omega_2\wedge\bar\omega_1)\,. \tag{3.5}$$

An alternative way to arrive at this Ansatz is to use the reparametrization invariance $u \to \tilde u(u,v,w)$ to arrange that the metric in the (u,φ_3) directions is proportional to $du^2 + u^2 d\varphi_3^2$. We may then use the reparametrization invariance in v and w to eliminate cross-terms of the form $du\,dv$ and $du\,dw$. Thus (3.4) provides the most general hermitian metric with the coordinates fixed in this manner. We have thus fixed the coordinates by prescribing the form of the metric, and we will solve for the functions that define the complex variables. As we will see, this inversion of the usual procedure leads to a significant simplifications.

To define the spinors, we introduce the frames:

$$\begin{aligned}
e^a \;&= H_0\, dx^a\,, \quad a = 1,\ldots,4\,, \qquad (e^5 + i\, e^{10}) \;=\; H_0^{-1}\, H_3\,\omega_3\,, \\
(e^6 + ie^9) \;&= H_0^{-1}\,(H_1\,\omega_1 \;+\; H_4\,\omega_2)\,, \qquad (e^7 + ie^8) \;=\; H_0^{-1}\, H_2\,\omega_2\,,
\end{aligned} \tag{3.6}$$

and thus:

$$A_0 \;=\; H_1\, H_4\,, \quad A_1 \;=\; H_1^2\,, \quad A_2 \;=\; (H_2^2 + H_4^2)\,, \quad A_3 \;=\; H_3^2\,. \tag{3.7}$$

3.2. The tensor gauge fields

Since we are dealing with a distribution of $D3$ branes, we define the five-form field strength in terms of a single potential function:

$$C_{(4)} \;=\; k\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,, \tag{3.8}$$

for some function, $k(u,v,w)$, and then take the five-form field strength to be:

$$F_{(5)} \;=\; dC_{(4)} + *dC_{(4)} \,. \tag{3.9}$$

The Ansatz for the two-form potential is simply to take the most general $(2,0)$ form with the appropriate phase dependence:

$$B_{(2)} \;=\; -i\, e^{i(\varphi_1+\varphi_2+\varphi_3)} \big[\, b_1\, \omega_2 \wedge \omega_3 \;-\; b_2\, \omega_1 \wedge \omega_3 \;+\; b_3\, \omega_1 \wedge \omega_2 \,\big] \,, \tag{3.10}$$

where the b_j are arbitrary functions of (u,v,w). In principle these functions could be complex, but they turn out to be real in the flow solution we find here.

3.3. The supersymmetries

It is convenient to define the supersymmetries via projection operators. In particular it is very useful to use projectors to isolate the supersymmetries that would be associated with $\mathcal{M}_6$ were it to be a Calabi-Yau manifold. To that end, define the projectors:

$$\Pi_0 \;=\; \frac{1}{2}\big[\, \mathbb{1} - i\, \Gamma^{1234} \,\big] \,, \qquad \Pi_1 \;=\; \frac{1}{2}\big[\, \mathbb{1} - i\, \Gamma^{78} \big] \,, \tag{3.11}$$

$$\Pi_2 \;=\; \frac{1}{2}\big[\, \mathbb{1} - i\, \Gamma^{69} \big] \,, \qquad \Pi_3 \;=\; \frac{1}{2}\big[\, \mathbb{1} + i\, \Gamma^{5\,10} \big] \,. \tag{3.12}$$

Define the spinor, ϵ_0, to be one that is constant and satisfies:

$$\Pi_j\, \epsilon_0 \;=\; 0 \,, \qquad j = 0,1,2,3 \,. \tag{3.13}$$

One of these projections is redundant because of the helicity condition: $\Gamma^{11}\epsilon = -\epsilon$ where $\Gamma^{11} \equiv \Gamma^{1\ldots 10}$.

Introduce the rotation matrix

$$\mathcal{O}(\beta) \;\equiv\; \cos(\tfrac{1}{2}\beta) + \sin(\tfrac{1}{2}\beta)\, \Gamma^{79} * \;. \tag{3.14}$$

where $*$ denotes the complex conjugation operator. The Killing spinor is then given explicitly by:

$$\epsilon \;=\; H_0^{\frac{1}{2}}\, e^{\frac{i}{2}(\varphi_1+\varphi_2+\varphi_3)}\,\mathcal{O}(\beta)\, e^{-i\varphi_3}\epsilon_0\,. \tag{3.15}$$

This spinor obeys the projection conditions:

$$\widehat{\Pi}_0\,\epsilon \;=\; 0\,, \qquad \Pi_1\,\epsilon \;=\; 0\,, \qquad \Pi_2\,\epsilon \;=\; 0\,, \tag{3.16}$$

where $\widehat{\Pi}_0$ is the dielectrically deformed projection operator [16–21]:

$$\widehat{\Pi}_0 \;=\; \frac{1}{2}\Big[\,\mathbb{1} - i\,\Gamma^{1234}\,\big(\cos(\beta) - e^{i(\varphi_1+\varphi_2+\varphi_3)}\sin(\beta)\,\Gamma^{79}\,*\,\big)\Big]\,, \tag{3.17}$$

Also observe that the spinor, ϵ, satisfies (2.11), (2.12) and (2.13), which, in fact, determine the dependence of ϵ upon the angles φ_j. The normalization factor, $H_0^{\frac{1}{2}}$, in (3.15) is fixed by the requirement that:

$$K^\mu \;=\; \bar\epsilon\,\Gamma^\mu\epsilon\,, \tag{3.18}$$

is a Killing vector.

4. Calabi-Yau Conditions

It is very instructive to look at the the conditions on the metric (3.4) and complex structure (3.5) required to make $\mathcal{M}_6$ into a Calabi-Yau space. This will not generate the flow background that we seek, but it will come very close.

4.1. Imposing the Kähler condition

It is convenient to introduce the matrices:

$$\mathcal{A} \;\equiv\; \begin{pmatrix} A_1 & A_0 \\ A_0 & A_2 \end{pmatrix}\,, \qquad \mathcal{H} \;\equiv\; \begin{pmatrix} v^{-1}\,\partial_v h_1 & w^{-1}\,\partial_w h_1 \\ v^{-1}\,\partial_v h_2 & w^{-1}\,\partial_w h_2 \end{pmatrix}\,, \tag{4.1}$$

and set

$$\mathcal{B} \;\equiv\; \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \;=\; \mathcal{A}\cdot\mathcal{H}\,. \tag{4.2}$$

Then the conditions that (3.5) be Kähler are equivalent to:

$$\partial_u\,\mathcal{B} \;=\; 0\,, \tag{4.3}$$

$$\tfrac{1}{w}\,\partial_w\,B_1 \;=\; \tfrac{1}{v}\,\partial_v\,B_2\,, \qquad \tfrac{1}{w}\,\partial_w\,B_3 \;=\; \tfrac{1}{v}\,\partial_v\,B_4\,, \tag{4.4}$$

and

$$\mathcal{A}\cdot\begin{pmatrix} u^{-1}\partial_u(u\,\partial_u h_1) \\ u^{-1}\partial_u(u\,\partial_u h_2)\end{pmatrix} \;=\; -\Big(\mathcal{H}^{-1}\Big)^t\cdot\begin{pmatrix} v^{-1}\,\partial_v\,A_3 \\ w^{-1}\,\partial_w\,A_3\end{pmatrix}\,, \tag{4.5}$$

where the superscript t denotes the transpose.

At large values of u we want the metric on $\mathcal{M}_6$ to become asymptotically flat:

$$ds_6^2 \to (du^2+u^2 d\varphi_3^2) \;+\; (dv^2+v^2 d\varphi_1^2) \;+\; (dw^2+w^2 d\varphi_2^2)\,, \tag{4.6}$$

and hence, at large u, we must have

$$h_1 \to \log(v)\,,\quad h_2 \to \log(w)\,,\qquad A_1 \to v^2\,,\quad A_2 \to w^2\,,\quad A_0 \to 0\,. \tag{4.7}$$

Therefore, for all values of u we must have:

$$\mathcal{B} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\,, \tag{4.8}$$

and thus:

$$\mathcal{A} \;\equiv\; \begin{pmatrix} A_1 & A_0 \\ A_0 & A_2\end{pmatrix} \;=\; \mathcal{H}^{-1} \;=\; \Delta^{-1}\begin{pmatrix} v\,\partial_w h_2 & -v\,\partial_w h_1 \\ -w\,\partial_v h_2 & w\,\partial_v h_1\end{pmatrix}\,, \tag{4.9}$$

where Δ is the Jacobian:

$$\Delta \;\equiv\; (\partial_v h_1)\,(\partial_w h_2) \;-\; (\partial_v h_2)\,(\partial_w h_1)\,. \tag{4.10}$$

This system of equations is elementary to analyze. First observe that (4.9) gives A_0, A_1 and A_2 in terms of h_j. Moreover, there are two equations for A_0 and these imply:

$$v\,\partial_w h_1 \;=\; w\,\partial_v h_2 \qquad\Leftrightarrow\qquad h_1 \;=\; \tfrac{1}{v}\,\partial_v g\,,\quad h_2 \;=\; \tfrac{1}{w}\,\partial_w g\,, \tag{4.11}$$

for some "master function," g. Finally, using $\mathcal{A}=\mathcal{H}^{-1}$ in (4.5) shows that:

$$A_3 \;=\; -u^{-1}\,\partial_u(u\,\partial_u g)\,. \tag{4.12}$$

Thus the coordinates and the entire Kähler metric are determined once we know the function g.

4.2. Calabi-Yau metrics

The following is a manifestly holomorphic $(3,0)$-form on $\mathcal{M}_6$:

$$\Omega \;\equiv\; dz_1 \wedge dz_2 \wedge dz_3 \;=\; e^{h_1+h_2}\, e^{i(\varphi_1+\varphi_2+\varphi_3)}\, \omega_1 \wedge \omega_2 \wedge \omega_3\,. \tag{4.13}$$

A Ricci-flat, Kähler metric has the property

$$\frac{1}{3!}\, J \wedge J \wedge J = \Omega \wedge \overline{\Omega}\,, \tag{4.14}$$

and using (3.5) and (4.13) this equation is equivalent to:

$$A_3\,(A_1\, A_2 - A_0^2) \;=\; e^{2(h_1+h_2)}\,. \tag{4.15}$$

Using the Kähler conditions derived above, this becomes:

$$\tfrac{1}{u}\,\partial_u(u\,\partial_u\, g) \;+\; \frac{\Delta}{v\,w}\,\exp\Big(2(\tfrac{1}{v}\,\partial_v g + \tfrac{1}{w}\,\partial_w g)\Big) \;=\; 0\,. \tag{4.16}$$

One can write this as a system for the h_j by acting with ∂_v and ∂_w:

$$\begin{aligned} \frac{1}{u}\,\partial_u(u\,\partial_u\, h_1) + \frac{1}{v}\,\partial_v\bigg(\frac{\Delta}{v\,w}\, e^{2(h_1+h_2)}\bigg) \;&=\; 0 \\ \frac{1}{u}\,\partial_u(u\,\partial_u\, h_2) + \frac{1}{w}\,\partial_w\bigg(\frac{\Delta}{v\,w}\, e^{2(h_1+h_2)}\bigg) \;&=\; 0\,, \end{aligned} \tag{4.17}$$

which is the natural generalization of the result presented in [7].

4.3. The Killing spinors on the Calabi-Yau manifold

Consider the spinor, $\hat\epsilon$, defined by taking $\beta = 0$ and $H_0 = 1$ in (3.15). This spinor obeys:

$$\frac{1}{2}\big[\,1\!\!1 + i\,\Gamma^{5\,10}\big]\,\hat\epsilon \;=\; \frac{1}{2}\big[\,1\!\!1 - i\,\Gamma^{69}\big]\,\hat\epsilon \;=\; \frac{1}{2}\big[\,1\!\!1 - i\,\Gamma^{78}\big]\,\hat\epsilon \;=\; 0\,, \tag{4.18}$$

and has a phase dependence:

$$\hat\epsilon \;=\; e^{\frac{i}{2}(\varphi_1+\varphi_2-\varphi_3)}\,\epsilon_0\,. \tag{4.19}$$

This spinor is, in fact, the one associated with the complex structure (3.5) but with $\varphi_3 \to -\varphi_3$. This change of complex structure is naturally suggested by the field theory

since it is consistent with the phases in (2.4). To be more explicit, consider a new set of holomorphic forms:

$$\begin{aligned}
\hat{\omega}_1 &\equiv dh_1 + i\, d\varphi_1 - (\partial_u h_1)\, \hat{\omega}_3 &=& \ (\partial_v h_1)\, dv + (\partial_w h_1)\, dw + i\,(d\varphi_1 + u\,\partial_u h_1\, d\varphi_3)\,, \\
\hat{\omega}_2 &\equiv dh_2 + i\, d\varphi_2 - (\partial_u h_2)\, \hat{\omega}_3 &=& \ (\partial_v h_2)\, dv + (\partial_w h_2)\, dw + i\,(d\varphi_2 + u\,\partial_u h_2\, d\varphi_3)\,, \\
\hat{\omega}_3 &\equiv du - i\, u\, d\varphi_3 = \bar{\omega}_3\,,
\end{aligned} \tag{4.20}$$

along with a complex structure defined by (3.5), but with $\omega_i \to \hat{\omega}_i$. The spinor, $\hat{\epsilon}$, is then the Killing spinor for the Calabi-Yau metric associated with this complex structure on $\mathcal{M}_6$. This change of complex structure generates some simple sign changes in the analysis above.

This observation about the complex structure will make no difference to the subsequent analysis in this paper, however it will prove important if one tries to to interpolate between the Calabi-Yau flow of [27] and that of [9,11] as discussed in [28,7]. In particular, it is important to note that the two-form tensor gauge fields are of type $(2,0)$ with respect to the complex structure (3.5), but in the $\beta \to 0$ limit the Killing spinor of that solution is not that of the Calabi-Yau space based upon the complex structure (3.5). Put differently, if one starts with the Calabi-Yau flow, then the B-field flux is *not* of type $(2,0)$ with respect to the complex structure of the Calabi-Yau metric: The holomorphic forms are complex conjugated in the (u, φ_3) direction.

5. The new flux solutions

The solutions with the non-trivial background fluxes closely parallel the Calabi-Yau solutions found in the previous section. The metric is no longer Kähler, but the two-form:

$$\widehat{J} \ \equiv\ A_1\, \omega_1 \wedge \bar{\omega}_1 \ +\ A_2\, \omega_2 \wedge \bar{\omega}_2 \ +\ A_3\, \cos(\beta)\, \omega_3 \wedge \bar{\omega}_3 \ +\ A_0\, (\omega_1 \wedge \bar{\omega}_2 \ +\ \omega_2 \wedge \bar{\omega}_1)\,, \tag{5.1}$$

is closed. In comparing this with (3.5), note the presence of the $\cos\beta$ in the third term. This means that $\hat{J}$ no longer yields an almost complex structure when combined with the metric (3.4), but the closure of $\widehat{J}$ provides a very convenient way of encoding some of the equations that define the new flux solutions. This means (4.3), (4.4) and (4.9) remain true, and that there is still a “master function,” g, defined by (4.11). Moreover, (4.5) is replaced by:

$$\mathcal{A} \cdot \begin{pmatrix} u^{-1}\partial_u(u\partial_u h_1) \\ u^{-1}\partial_u(u\partial_u h_2) \end{pmatrix} \ =\ \left(\mathcal{H}^{-1}\right)^t \cdot \begin{pmatrix} v^{-1}\, \partial_v\, (A_3 \cos(\beta)) \\ w^{-1}\, \partial_w\, (A_3 \cos(\beta)) \end{pmatrix} . \tag{5.2}$$

The sign change in (5.2) as compared to (4.5) is due to the change complex structure described in section 4. To specify A_3 and β independently, we need a further equation, and this is:

$$\mathcal{A}\cdot\begin{pmatrix} u^{-1}\partial_u h_1 \\ u^{-1}\partial_u h_2 \end{pmatrix} \;=\; \Big(\mathcal{H}^{-1}\Big)^t\cdot\begin{pmatrix} v^{-1}\,\partial_v\,\big(\frac{1}{2}A_3\cos^2(\frac{1}{2}\beta)\big) \\ w^{-1}\,\partial_w\,\big(\frac{1}{2}A_3\cos^2(\frac{1}{2}\beta)\big)\big) \end{pmatrix}\,. \tag{5.3}$$

Combining this with (5.2) one obtains:

$$\mathcal{A}\cdot\begin{pmatrix} u^3\partial_u(u^{-3}\partial_u h_1) \\ u^3\partial_u(u^{-3}\partial_u h_2) \end{pmatrix} \;=\; -\Big(\mathcal{H}^{-1}\Big)^t\cdot\begin{pmatrix} v^{-1}\,\partial_v\,A_3 \\ w^{-1}\,\partial_w\,A_3 \end{pmatrix}\,. \tag{5.4}$$

Using (4.9) and (4.11), and particularly the fact that $\mathcal{A}=\mathcal{H}^{-1}$, one can integrate these equations:

$$A_3 \;=\; -u^3\partial_u(u^{-3}\,\partial_u g) \;+\; k_1(u)\,, \qquad \tfrac{1}{2}A_3\cos^2(\tfrac{1}{2}\beta) \;=\; u^{-1}\partial_u g \;+\; k_2(u)\,, \tag{5.5}$$

where k_1, k_2 are arbitrary functions of u. A more detailed analysis of the supersymmetry variations, yields:

$$A_3 \;=\; -u^3\,\partial_u\big(\tfrac{1}{2}\,u^{-2}\,A_3\cos^2(\tfrac{1}{2}\beta)\big)\,, \tag{5.6}$$

from which we obtain $k_1(u)=-u^3\partial_u(u^{-2}k_2(u))$. To fix these functions completely one should first note that the function g is itself only defined by (4.11) up to an arbitrary function of u, and so we can set $k_2=0$ by absorbing it into the definition of g. It follows that $k_1=0$. Finally, note that at large v,w we must have $A_3\to 1$ and $\beta\to 0$, and so we must have

$$g(u,v,w) \;\sim\; \tfrac{1}{4}\,u^2\,, \qquad v,w\to\infty\,. \tag{5.7}$$

The supersymmetry also requires (4.14) and hence (4.15). Thus we have the following differential equation for g:

$$u^3\,\partial_u(u^{-3}\,\partial_u\,g) \;+\; \frac{\Delta}{v\,w}\,\exp\Big(2(\tfrac{1}{v}\,\partial_v g+\tfrac{1}{w}\,\partial_w g)\Big) \;=\; 0\,, \tag{5.8}$$

which can also be converted to system for the h_j by acting with ∂_v and ∂_w:

$$\begin{aligned} u^3\,\partial_u\Big(\frac{1}{u^3}\,\partial_u\,h_1\Big) + \frac{1}{v}\,\partial_v\bigg(\frac{\Delta}{v\,w}\,e^{2(h_1+h_2)}\bigg) \;&=\; 0 \\ u^3\,\partial_u\Big(\frac{1}{u^3}\,\partial_u\,h_2\Big) + \frac{1}{w}\,\partial_w\bigg(\frac{\Delta}{v\,w}\,e^{2(h_1+h_2)}\bigg) \;&=\; 0\,, \end{aligned} \tag{5.9}$$

Using (4.7) and (5.7) we find that g must satisfy (5.8) and with boundary conditions:

$$g(u,v,w) \ \sim \ \tfrac{1}{4}\big[\, u^2 \ + \ v^2\,(2\,\log(v)-1) \ + \ w^2\,(2\,\log(w)-1)\big]\,, \quad u,v,w\to\infty\,. \qquad (5.10)$$

Once one finds a solution to this equation one determines the metric functions on $\mathcal{M}_6$ via (4.9) and (5.5), which we summarize as:

$$\begin{aligned} A_1 &= \frac{v}{w\,\Delta}\,\partial_w^2 g\,, \qquad & A_2 &= \frac{w}{v\,\Delta}\,\partial_v^2 g\,, \\ A_0 &= \frac{1}{\Delta}\,\partial_v\partial_w g\,, \qquad & A_3 &= -u^3\partial_u(u^{-3}\,\partial_u g)\,. \end{aligned} \qquad (5.11)$$

The deformation angle, β, is given by (5.5), and this may be rewritten as

$$\cos^2(\tfrac{1}{2}\beta) \ = \ -\frac{2\,\partial_u g}{u^4\partial_u(u^{-3}\,\partial_u g)}\,. \qquad (5.12)$$

The remaining parts of the solution are simple to determine from the functions defined above. Exactly as in [19], we have:

$$H_0^2 \ = \ \frac{a\,u}{\sqrt{A_3}\,\sin\beta}\,, \qquad k \ = \ -\tfrac{1}{4}\,H_0^4\,\cos\beta\,, \qquad (5.13)$$

where a is a constant of integration, and k is the function in (3.8). The constant, a, may be absorbed into a rescaling of the coordinates, but it is often convenient to retain it. Finally, the two-form flux functions, b_j, are given by:

$$b_1 \ = \ \tfrac{2}{a\,u}\,e^{h_1+h_2}\,\partial_u h_1\,, \qquad b_2 \ = \ \tfrac{2}{a\,u}\,e^{h_1+h_2}\,\partial_u h_2\,, \qquad b_3 \ = \ -\tfrac{2}{a\,u}\,\sin^2(\tfrac{1}{2}\beta)\,e^{h_1+h_2}\,. \qquad (5.14)$$

One can re-write the fluxes in a slightly more compact form using (3.2) and (3.10):

$$B_{(2)} \ = \ \frac{2i}{\bar z_3}\,\big[\, dz_1\wedge dz_2 \ - \ z_1\,z_2\,\cos^2(\tfrac{1}{2}\beta)\,\omega_1\wedge\omega_2\,\big]\,. \qquad (5.15)$$

It is also worth noting that if we convert this to frames then $B_{(2)}$ has the following component in the 6789 direction:

$$a\,i\,\tan(\tfrac{1}{2}\beta)\,e^{i(\varphi_1+\varphi_2+\varphi_3)}\,(e^6+ie^9)\wedge(e^7+ie^8)\,, \qquad (5.16)$$

exactly as in [19].

Thus, we once again find a family of solutions that are almost Calabi-Yau in that the metric is hermitian with respect to an integrable complex structure, and the holomorphic $(3,0)$ form satisfies (4.14). The metric is simply not Kähler, and there is a non-trivial flux that can be arranged to have a potential of $(2,0)$ type. The underlying master differential equation that governs our new solution is a relatively simple deformation of the equation that governs the corresponding Calabi-Yau metrics, and follows a very similar pattern to that obtained in [19].

6. Radial Coulomb branch flows

The geometry of $\mathcal{N}=1$ supersymmetric Coulomb branch flows in which the branes are allowed to spread purely *radially* were analyzed in [19,7]. These flows preserve the $SU(2)$ global symmetry that rotates (Φ_1, Φ_2) as a doublet. Accordingly, the complex coordinates analogous to (2.4) were taken to be:

$$\Phi_1 \sim V\ \cos(\tfrac{1}{2}\phi_1)\, e^{-\frac{i}{2}\,(\phi_2+\phi_3)}\,, \qquad \Phi_2 \sim V\ \sin(\tfrac{1}{2}\phi_1)\, e^{\frac{i}{2}\,(\phi_2-\phi_3)}\,, \qquad \Phi_3 \sim u\, e^{i\phi}\,. \tag{6.1}$$

By comparing this to (2.4) we see that the relevant change of coordinates is:

$$v \ = \ V\cos(\tfrac{1}{2}\,\phi_1) \qquad w = V\sin(\tfrac{1}{2}\,\phi_1)\,, \qquad \varphi_1 = \tfrac{1}{2}(\phi_3+\phi_2)\,, \qquad \varphi_2 = \tfrac{1}{2}(\phi_3-\phi_2)\,. \tag{6.2}$$

Similarly, the analog of (3.2) was:

$$z_1 \ = \ e^{\frac{1}{2}\,\Psi}\ \cos(\tfrac{1}{2}\phi_1)\, e^{\frac{i}{2}\,(\phi_2+\phi_3)}\,, \quad z_2 \ = \ e^{\frac{1}{2}\,\Psi}\ \sin(\tfrac{1}{2}\phi_1)\, e^{-\frac{i}{2}\,(\phi_2-\phi_3)}\,, \qquad z_3 \ = \ u\, e^{-i\,\phi}\,. \tag{6.3}$$

and hence one has:

$$h_1 = \tfrac{1}{2}\,\Psi \ + \ \log\cos(\tfrac{1}{2}\,\phi_1) \qquad h_2 = \tfrac{1}{2}\,\Psi \ + \ \log\sin(\tfrac{1}{2}\,\phi_1)\,. \tag{6.4}$$

One can then easily show that:

$$\frac{\Delta}{v\,w}\, e^{2(h_1+h_2)} \ = \ \frac{e^{2\Psi}}{2V^3}\,\frac{\partial\Psi}{\partial V}\,, \tag{6.5}$$

and from this it follows that both equations in (4.17) reduce to:

$$\tfrac{1}{u}\,\partial_u\big(u\,\partial_u\Psi\big) \ + \ \tfrac{1}{V}\,\partial_V\big(\tfrac{1}{V^3}\,e^{2\Psi}\,\partial_V\Psi\big) \ = \ 0\,, \tag{6.6}$$

while both equations in (5.9) reduce to:

$$u^3\,\partial_u\big(\tfrac{1}{u^3}\,\partial_u\Psi\big) \ + \ \tfrac{1}{V}\,\partial_V\big(\tfrac{1}{V^3}\,e^{2\Psi}\,\partial_V\Psi\big) \ = \ 0\,. \tag{6.7}$$

This exactly reproduces the results of [19].

One should also note that to completely solve the flow solution in [19] it required an auxiliary function defined by:

$$\frac{\partial\mathcal{S}}{\partial u} \ = \ -\frac{1}{2\,u^3\,V^3}\,\frac{\partial e^{2\Psi}}{\partial V}\,, \qquad \frac{\partial\mathcal{S}}{\partial V} \ = \ \frac{V}{u^3}\,\frac{\partial\Psi}{\partial u}\,. \tag{6.8}$$

If one uses (6.5), (5.8) and (4.11) then one finds that:

$$\mathcal{S} \;=\; \tfrac{2}{u^3}\,\partial_u g\,. \tag{6.9}$$

In other words, in [19] one could have viewed the second equation in (6.8) as implying the existence of a function, g, with:

$$\mathcal{S} \;=\; \tfrac{2}{u^3}\,\partial_u g\,, \qquad \Psi \;=\; \tfrac{2}{V}\,\partial_V g\,, \tag{6.10}$$

and then the first equation in (6.8) becomes a differential equation for g, and this is simply the reduction of (5.8). Thus, by recasting the entire problem in terms of g, one sees that there is nothing in the work of [19] that is special to two variables: All the interesting functions merely emerge as partial derivatives of the single, underlying master function, g.

7. Gauged supergravity flows

There is a three-parameter family of solutions that should be among the new flux solutions presented in section 5. This family was obtained by "lifting" solutions of five-dimensional, gauged supergravity [14,15], and we now show how it fits into our more general class of solutions. We will first summarize the details of the original gauged supergravity solution and identify its integrable complex structure by giving the holomorphic $(1,0)$-forms. We then re-write the metric in terms of these forms as in (3.4) and identify the functions, $A_0, \ldots, A_3$. Having done this we can read off the master functions, h_1 and h_2, and then verify that they determine all the other functions in the solution, as outlined in section 5. In particular, one can verify that that the equations of motion in five-dimensional supergravity imply that the master equations, (5.9), are satisfied.

7.1. The known solution and its holomorphic structure

In five-dimensions this solution is characterized in terms of three scalar fields, denoted by χ, ν and ρ, and a superpotential, W. The superpotential is [14,15]:

$$W \;=\; \frac{1}{4}\,\rho^4\,\big(\cosh(2\chi) - 3\big) \;-\; \frac{1}{4\,\rho^2}\,(\nu^2 + \nu^{-2})\,\big(\cosh(2\chi) + 1\big)\,, \tag{7.1}$$

and the equations of motion are:

$$\frac{d\rho}{dr} \;=\; \frac{1}{6L}\,\rho^2\,\frac{\partial W}{\partial \rho}\,, \qquad \frac{d\nu}{dr} \;=\; \frac{1}{2L}\,\nu^2\,\frac{\partial W}{\partial \nu}\,, \qquad \frac{d\chi}{dr} \;=\; \frac{1}{L}\,\frac{\partial W}{\partial \chi}\,. \tag{7.2}$$

The five-dimensional metric is then given by

$$ds_{1,4}^2 \;=\; dr^2 \;+\; e^{2A(r)} \big(\, \eta_{\mu\nu}\, dx^\mu\, dx^\nu \big)\,, \tag{7.3}$$

where

$$\frac{dA}{dr} \;=\; -\frac{2}{3\,L}\, W\,. \tag{7.4}$$

To lift this to ten dimensions the authors of [14,15] introduced the following coordinates to parametrize the five-sphere in $\mathbb{C}^3$:

$$\begin{aligned} z_1 \;\equiv x_1 + i\, x_2 \;&=\; \cos\theta\, \cos\phi\, e^{i\,\varphi_1}\,, \quad z_2 \;\equiv\; x_3 - i\, x_4 \;=\; \cos\theta\, \sin\phi\, e^{-i\,\varphi_2}\,, \\ z_3 \;\equiv x_5 - i\, x_6 \;&=\; \sin\theta\, e^{-i\,\varphi_3}\,. \end{aligned} \tag{7.5}$$

The five-dimensional solution then lifts to the ten-dimensional metric:

$$ds_{10}^2 \;=\; \Omega^2\, ds_{1,4}^2 \;+\; ds_5^2\,, \tag{7.6}$$

where Ω, is given by:

$$\Omega \;\equiv\; (\cosh\chi)^{\frac{1}{2}} \big(\rho^{-2}\, (\nu^2 \cos^2\phi + \nu^{-2} \sin^2\phi)\, \cos^2\theta \;+\; \rho^4\, \sin^2\theta\big)^{\frac{1}{4}}\,. \tag{7.7}$$

The metric, ds_5^2, is a complicated metric on the deformed five-sphere:

$$\begin{aligned} ds_5^2 \;=\; & L^2\, \Omega^{-2} \Big[\rho^{-4} \big(\cos^2\theta + \rho^6\, \sin^2\theta\, (\nu^{-2} \cos^2\phi + \nu^2\, \sin^2\phi)\big)\, d\theta^2 \\ & + \rho^2 \cos^2\theta\, (\nu^2\, \cos^2\phi + \nu^{-2} \sin^2\phi)\, d\phi^2 \\ & - 2\, \rho^2\, (\nu^2 - \nu^{-2})\, \sin\theta\, \cos\theta\, \sin\phi\, \cos\phi\, d\theta\, d\phi \\ & + \; \rho^2\, \cos^2\theta\, (\nu^{-2} \cos^2\phi\, d\varphi_1^2 + \nu^2\, \sin^2\phi\, d\varphi_2^2) \;+\; \rho^{-4}\, \sin^2\theta\, d\varphi_3^2\Big] \\ & + L^2\, \Omega^{-6}\, \sinh^2\chi\, \cosh^2\chi\, \big(\cos^2\theta\, (\cos^2\phi\, d\varphi_1 - \sin^2\phi\, d\varphi_2) - \sin^2\theta\, d\varphi_3\,\big)^2\,. \end{aligned} \tag{7.8}$$

where L is the radius of the round sphere.

To relate this to the solution presented here one makes the change of variable:

$$u \;=\; e^{\frac{3}{2}\,A} \sqrt{\sinh\chi}\, \sin\theta\,, \quad v \;=\; e^A \rho\, \nu^{-1} \cos\theta \cos\phi\,, \quad w \;=\; e^A \rho\, \nu \cos\theta \sin\phi\,. \tag{7.9}$$

The metric (7.8) can now be written in terms of the (integrable) holomorphic forms:

$$\begin{aligned} \omega_3 &\equiv du \;-\; i\, u\, d\varphi_3\,, \\ \omega_1 &\equiv dv \;-\; v\, \mu_1 \;+\; i\, v\, d\varphi_1 \;+\; \frac{v}{u} \Big(\frac{\nu^2 \sinh^2\chi\, \sin^2\theta}{X_0}\Big)\, \omega_3\,, \\ \omega_2 &\equiv dw \;-\; w\, \mu_2 \;+\; i\, w\, d\varphi_2 \;+\; \frac{w}{u} \Big(\frac{\nu^{-2} \sinh^2\chi\, \sin^2\theta}{X_0}\Big)\, \omega_3\,, \end{aligned} \tag{7.10}$$

where

$$\mu_1 \equiv \frac{\nu^2 s^2}{L\rho^2} dr\,, \qquad \mu_2 \equiv \frac{\nu^{-2} s^2}{L\rho^2} dr\,,$$
$$X_0 \equiv \rho^6 \sin^2\theta \;+\; c^2\, \cos^2\theta \left(\nu^2 \cos^2\phi \;+\; \nu^{-2} \sin^2\phi\right), \tag{7.11}$$

and we have adopted the convenient shorthand:

$$c \equiv \cosh\chi\,, \qquad s \equiv \sinh\chi\,. \tag{7.12}$$

The six-dimensional metric metric that underlies (7.6) is:

$$ds_6^2 \;=\; L^{-2}\, e^{2\,A} \left(\Omega^4\, dr^2 \;+\; \Omega^2\, ds_5^2\right), \tag{7.13}$$

has precisely the form (3.4) with

$$\begin{aligned} A_1 &= Y_0^{-1}(\rho^6 \sin^2\theta + \cos^2\theta(c^2\,\nu^2 \cos^2\phi + \nu^{-2}\sin^2\phi))\,, \\ A_2 &= Y_0^{-1}(\rho^6 \sin^2\theta + \cos^2\theta(\nu^2 \cos^2\phi + c^2\,\nu^{-2}\sin^2\phi))\,, \\ A_3 &= \frac{e^{2\,A}\, c^2\, \sin^2\theta}{\rho^4\, u^2\, X_0}\, Y_0\,, \qquad A_0 \;=\; \frac{e^{-2\,A}\, s^2\, v\, w}{\rho^2\, Y_0}\,, \end{aligned} \tag{7.14}$$

where

$$Y_0 \;\equiv\; \rho^6 \sin^2\theta + \cos^2\theta\,(\nu^2 \cos^2\phi + \nu^{-2}\sin^2\phi)\,. \tag{7.15}$$

To verify this one first uses the co-ordinate transformations in (7.9) to obtain the holomorphic forms, ω_i, of (7.10) in terms of the co-ordinates (r,θ,ϕ). One can then use these expressions along with metric functions, A_i, given in (7.14) to obtain, after a rather long but straightforward computation, the metric in (7.8).

7.2. The master functions

By comparing (4.20) and (7.10), we can read off the exterior derivatives of the functions, h_j:

$$d\,h_1 \;=\; \frac{dv}{v} \;-\; \mu_1\,, \qquad d\,h_2 \;=\; \frac{dw}{w} \;-\; \mu_2\,. \tag{7.16}$$

One also finds the conditions:

$$\partial_u h_1 \;=\; \Big(\frac{\nu^2 \sinh^2\chi\, \sin^2\theta}{u\, X_0}\Big), \qquad \partial_u h_2 \;=\; \Big(\frac{\nu^{-2} \sinh^2\chi\, \sin^2\theta}{u\, X_0}\Big). \tag{7.17}$$

One can easily check that this is consistent with (7.16), indeed, using the change of variables (7.9) one can check that:

$$\begin{aligned} \mu_1 &= \frac{e^{-2A}\, s^2}{\rho^2\, X_0}\big(\nu^4\, v dv \;+\; w dw\big) \;+\; \frac{\nu^2\, s^2\, \sin^2\theta}{u\, X_0}\, du\,, \\ \mu_2 &= \frac{e^{-2A}\, s^2}{\rho^2\, X_0}\big(v dv \;+\; \nu^{-4}\, w dw\big) \;+\; \frac{s^2\, \sin^2\theta}{\nu^2\, u\, X_0}\, du\,. \end{aligned} \tag{7.18}$$

Define functions:

$$q_1 \;\equiv\; \int \frac{\nu^2 s^2}{L\rho^2}\, dr\,, \qquad q_2 \;\equiv\; \int \frac{\nu^{-2} s^2}{L\rho^2}\, dr\,, \tag{7.19}$$

and then we have

$$h_1 \;=\; \log(v) \;-\; q_1\,, \qquad h_2 \;=\; \log(w) \;-\; q_2\,. \tag{7.20}$$

While we do not know explicit expressions for these function individually one can easily show that:

$$e^{2\,(h_1+h_2)} \;=\; v^2\, w^2\, e^{-A}\, \rho^{-4}\, c^2\, s^{-1}\,, \tag{7.21}$$

which greatly simplifies (5.9).

Note that apart from the trivial log terms, the functions, h_j are functions of only the original radial coordinate in anti-de Sitter space. We therefore have the h_j implicitly in terms of u, v and w. One can use the equations of motion (7.2) to verify that h_1 and h_2 satisfy (5.9).

One can now obtain the function, g, by integrating:

$$h_1 \;=\; \frac{1}{v}\,\frac{\partial g}{\partial v}\,, \qquad h_2 \;=\; \frac{1}{w}\,\frac{\partial g}{\partial w}\,, \qquad A_3 \;=\; -u^3\,\frac{\partial}{\partial u}\Big(\frac{1}{u^3}\,\frac{\partial g}{\partial u}\Big)\,. \tag{7.22}$$

It is trivial to integrate the log terms to obtain:

$$g \;=\; \tfrac{1}{4}\,(v^2 \;+\; w^2) \;+\; \tfrac{1}{2}\, v^2\, \log(v) \;+\; \tfrac{1}{2}\, w^2\, \log(w) \;+\; \tilde g\,, \tag{7.23}$$

where

$$\tfrac{1}{v}\,\partial_v \tilde g \;=\; -q_1\,, \qquad \tfrac{1}{w}\,\partial_w \tilde g \;=\; -q_2\,, \qquad A_3 \;=\; -u^3\,\frac{\partial}{\partial u}\Big(\frac{1}{u^3}\,\frac{\partial \tilde g}{\partial u}\Big)\,. \tag{7.24}$$

Since the q_j are only known explicitly as functions of r, and thus implicitly as a function of (u, v, w), one can, at best hope to determine $\tilde g$ as a function of r, θ and ϕ. One can make significant progress in doing this explicitly, and in this is described in the Appendix. We have shown here exactly how the solution of [14,15] appears as a special solution to the general class of solutions derived in section 5.

8. Conclusions

We have defined a large class of $\mathcal{N} = 1$ supersymmetric holographic flow solutions, and reduced them to finding a "master function" that is the solution of a single partial differential equation. As has been observed in other papers [17–20], such differential equations naturally linearize at infinity and have a straightforward perturbation expansion. More significantly, this equation is once again a rather simple deformation of the Calabi-Yau condition. Indeed, the entire geometry is, once again, almost Calabi-Yau in that it has an integrable complex structure with a hermitian metric. There is also a holomorphic $(3,0)$ form, Ω, such that $\Omega \wedge \bar{\Omega}$ is the volume form. The crucial difference between our solution and a Calabi-Yau compactification is that the metric is not Kähler, and there is a non-trivial, non-normalizable 3-form flux. This flux dielectrically polarizes the $D3$-branes into $D5$-branes [29] and this is reflected in the deformation of one of the projectors that defines the supersymmetry [16–21].

Our results here represent a significant extension of the results of [19]. On a technical level we have found a class of solutions with a smaller amount of symmetry, in which the initial Ansatz is based upon functions of three variables. On a more physical level, we have a family of flows that probes *two* independent directions of the Coulomb branch of the non-trivial $\mathcal{N} = 1$ supersymmetric fixed-point. That is, our solutions describe brane configurations that can spread in independent radial distributions in each of the two complex directions of the Coulomb branch of the $\mathcal{N} = 1$ supersymmetric flows. We have shown

While we have completely characterized our solutions via a simple deformation of the Calabi-Yau conditions, this deformation is expressed rather technically in terms of changing coefficients of the master differential equation. This must have some more natural geometric interpretation. The results presented here highlight the strong connections to Calabi-Yau geometry, and as a result will provide a very useful basis for investigating the deformation of the geometry. Work on this is proceeding.

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Appendix A. Finding the master function of the gauged supergravity flow

One can make significant progress in partially integrating (7.24) to obtain $\tilde{g}(r,\theta,\phi)$. The first step is to write these equations in terms of derivatives with respect to r,θ and ϕ. The first two equations in (7.24) reduce to:

$$\partial_\phi \tilde{g} \ = \ e^{2\,A}\,\rho^2\,(q_1\,\nu^{-2} - q_2\,\nu^2)\,\cos^2\theta\,\sin\phi\,\cos\phi\,, \tag{A.1}$$

$$\begin{aligned}(\partial_r - (\partial_r \log(u))\,\tan\theta\,\partial_\theta)\,\tilde{g} \ &= \ -L^{-1}\,e^{2\,A}\Big[c^2\,(q_1\,\cos^2\phi \ + \ q_2\,\sin^2\phi)\,\cos^2\theta \\ &\qquad + \ \rho^6\,(\nu^{-2}\,q_1\,\cos^2\phi \ + \ \nu^2\,q_2\,\sin^2\phi)\,\sin^2\theta\Big]\,.\end{aligned} \tag{A.2}$$

where $u(r,\theta)$ is defined by (7.9). Observe that the differential operator on the left-hand side of (A.2) annihilates the coordinate, u, and so these two equations indeed only give information about the v and w dependence of $\tilde{g}$.

Equation (A.1) is trivial to integrate and yields

$$\tilde{g} = -\tfrac{1}{4}\,e^{2\,A}\,\rho^2\,(q_1\,\nu^{-2} - q_2\,\nu^2)\,\cos^2\theta\,\cos(2\,\phi) \ + \ p(r,\theta)\,, \tag{A.3}$$

for some function, $p(r,\theta)$. One can now substitute this into (A.2) to obtain an equation for $p(r,\theta)$.

At this point it is convenient to introduce a change of variables:

$$\begin{aligned} z \ &\equiv \ \tfrac{1}{2}\,\log(u) \ = \ \tfrac{1}{4}\,\log\Big(e^{3\,A}\sinh\chi\,\sin^2\theta\Big)\,, \\ t \ &\equiv \ \tfrac{1}{2}\,\log\Big(\frac{u}{\sin^2\theta}\Big) \ = \ \tfrac{1}{4}\,\log\Big(\frac{e^{3\,A}\sinh\chi}{\sin^2\theta}\Big)\,. \end{aligned} \tag{A.4}$$

We then find:

$$\partial_t\,p \ = \ -\tfrac{1}{2}\,e^{2\,A}\,\rho^{-4}\Big[c^2\,(q_1+q_2)\,(1-\sin^2\theta) \ + \ \rho^6\,(\nu^{-2}\,q_1 + \nu^2\,q_2)\,\sin^2\theta\Big]\,. \tag{A.5}$$

Note that $\sin\theta = e^{z-t}$ while e^{z+t} is purely a function of r, and so (A.5) has the form:

$$\partial_t\,p \ = \ f_1(z+t) \ + \ e^{4z}\,f_2(z+t)\,, \tag{A.6}$$

for some functions, f_1 and f_2. This is trivially solved by quadrature, and the result is:

$$\begin{aligned} p(r,\theta) \ &= \ r(u) \ - \ \tfrac{1}{2\,L}\int dr\,e^{2A}\,c^2\,(q_1+q_2) \\ &\quad + \ \tfrac{1}{2\,L}\,u^2\int dr\,e^{-A}\,s^{-1}\,\Big(c^2\,(q_1+q_2) \ - \ \rho^6\,(\nu^{-2}\,q_1 + \nu^2\,q_2)\Big)\,. \end{aligned} \tag{A.7}$$

where $r(u)$ is some, as yet, arbitrary function of u, and we have used the fact that $e^{4z} = u^2$. The function, $r(u)$, can then be determined by substituting (A.3) and (A.7) into the third equation in (7.24).

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